Unsteady electrohydrodynamic flow of immiscible liquids in channels

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An unsteady electrohydrodynamic flow of two viscous immiscible liquids in a channel is theoretically investigated. The flow is caused by an oscillating electric field whose intensity vector has both normal and tangential components on the interface between the liquids. The velocity profile and flow rates of the liquids are found for steady-state oscillatory flows. The limiting cases of constant and high-frequency fields are discussed.

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I. INTRODUCTION

Enormous recent attention to microfluidics [1,2] revived interest to various phenomena that can be used for actuation of fluids in microchannels. Actuation by electric fields seems very promising since it allows integrating pumps without moving parts into microfluidic devices [2]. Electric fields cause electro-osmotic and electrohydrodynamic (EHD) flows due to action of the Coulomb force on uncompensated electric charges contained in a fluid or on its boundaries [2,3]. Electro-osmotic flows take place usually in electrolytes, in which uncompensated charges appear due to chemical interaction of the electrolyte with a solid boundary and form electric double layer. EHD flows occur in leaky dielectrics, where uncompensated charges are either injected from electrodes or induced by the electric field due inhomogeneity of the electric conductivity and dielectric permittivity. The comparative analysis of the models that are used for theoretical investigation of electro-osmotic and EHD flows is given by Saville's review [3].

In particular, EHD flow arises when the intensity vector of an electric field on the interface between two immiscible liquids has both normal and tangential components and the ratio of the dielectric permittivities of the liquids does not equal that of electric conductivities [3,4]. The scope of the present work is restricted by the study of this particular case of EHD flows. As an example of such a flow, Melcher and Taylor considered a steady EHD flow of a liquid layer with free boundary in their review [4]. In the present work, this example is generalized in order to make it more applicable to microfluidics. Since liquids with common interfaces are more frequently encountered in microfluidic devices than those with free boundaries, a flow of two immiscible liquid layers in a channel is considered. Work of microfluidic devices that use integrated EHD pumps implies turning on and off the electric field or varying its intensity, i.e., forcing transient processes. Thus, further improvement of such devices requires the study of transient processes in them, in particular, investigation of unsteady EHD flows. So an unsteady EHD flow of the layers caused by a harmonically oscillating electric field is considered. (Melcher and Taylor's example is the limiting case as the viscosity of one of the liquids and the frequency of oscillations tend to zero.)

The setting of the problem, the system of equations, and boundary conditions are provided in Sec. II. In Sec. III, the problem is solved, the solution is written down, and its limiting cases are considered. The obtained results are discussed in Sec. IV.

II. SETTING OF THE PROBLEM

Consider two immiscible liquids in a channel of the length L (see Fig. 1). The cross section of the channel is a rectangle with width D and height h such that $h \ll D \ll L$ providing that the influence of the lateral walls and of the flow in the inlet and outlet of the channel on the velocity profile that is established along the largest part of its length can be neglected. The interface between the liquids is plane so that the liquids occupy the layers of heights h_1 and h_2 ($h=h_1+h_2$). The electric conductivities, dielectric permittivities, and viscosities of the liquids are $\lambda_1, \varepsilon_1, \eta_1$ and $\lambda_2, \varepsilon_2, \eta_2$, respectively. The electric conductivities of the liquids are so small that the conditions for EHD approximation [4] is satisfied. The upper and lower walls of the channel are electrodes to which alternate transversal and longitudinal voltages

$$V_{\mathrm{t}}(t) = V_{\mathrm{ta}} \cos(\omega t), \quad V_{\mathrm{l}}(t) = V_{\mathrm{la}} \cos(\omega t) \tag{1}$$

are applied (t is the time and ω is the angular frequency). The transversal voltage provides the normal component of the electric intensity vector, which induces uncompensated charges on the interface; and the longitudinal voltage creates the tangential component, which acts on the charges and actuates the liquids.

A. Equations

Provided that the liquids remain electroneutral in the bulk at all times (and that the uncompensated charge is localized

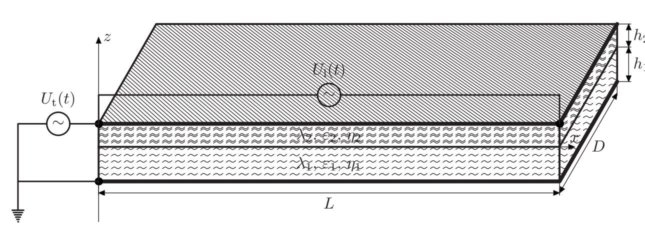

FIG. 1. Setting of the problem.

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in the infinitely thin interface), Maxwell's equations for the electric field in the EHD approximation, the electric charge conservation law, and constitutive relations take the form [4]

$$\vec{\nabla} \cdot \vec{D} = 0, \quad \vec{\nabla} \times \vec{E} = \vec{0}, \tag{2}$$

$$\vec{\nabla} \cdot \vec{j} = 0, \tag{3}$$

$$\vec{D} = \varepsilon \vec{E}, \quad \vec{j} = \lambda \vec{E}. \tag{4}$$

The equations for the flow include the momentum conservation law and the continuity equation for incompressible fluids (cf. [4]),

$$\rho \frac{\partial \vec{v}}{\partial t} + \rho \vec{v} \cdot \vec{\nabla} \vec{v} = -\vec{\nabla} p + \vec{\nabla} \cdot \hat{\sigma}_{\mathrm{v}} + \vec{\nabla} \cdot \hat{\sigma}_{\mathrm{e}}, \tag{5}$$

$$\vec{\nabla} \cdot \vec{v} = 0, \tag{6}$$

where $\hat{\sigma}_{\mathrm{v}}$ and $\hat{\sigma}_{\mathrm{e}}$ are the viscous and Maxwell's stress tensors

$$\hat{\sigma}_{\mathrm{v}} = 2\eta (\vec{\nabla} \vec{v})^{\mathrm{S}}, \tag{7}$$

$$\hat{\sigma}_{\mathrm{e}} = \frac{1}{4\pi} \vec{D}\ \vec{E} - \frac{1}{8\pi} \vec{D} \cdot \vec{E}. \tag{8}$$

Using Eqs. (2), (4), and (6), and constancy of ε and η, one obtains the Navier-Stokes equation for the flows within each layer,

$$\rho \frac{\partial \vec{v}}{\partial t} + \rho \vec{v} \cdot \vec{\nabla} \vec{v} = -\vec{\nabla} p + \eta \Delta \vec{v}. \tag{9}$$

Maxwell's equations (2) can be replaced with Laplace's equation for the electric potential φ ($\vec{E} = -\vec{\nabla} \varphi$),

$$\Delta \varphi = 0, \tag{10}$$

with the electric charge conservation law (3) being identically satisfied. Here, $\vec{E}$ and $\vec{D}$ are the intensity and induction of the electric field; $\vec{j}$ is the electric current density; $\vec{v}$ and p are the velocity and pressure; $\varepsilon = \varepsilon_1$, $\lambda = \lambda_1$, $\eta = \eta_1$, and $\varepsilon = \varepsilon_2$, $\lambda = \lambda_2$, $\eta = \eta_2$ within the corresponding layer; $\vec{\nabla}$ and Δ denote the nabla operator and Laplacian; $\hat{T}^{\mathrm{S}}$ denotes the symmetric part of tensor $\hat{T}$; and $\vec{a}\vec{b}$, $\vec{a} \cdot \vec{b}$, and $\vec{a} \times \vec{b}$ denote the dyadic, scalar, and vector products of vectors $\vec{a}$ and $\vec{b}$. Here and in what follows, all the formulas are written down for the Gaussian system of units.

Choose a Cartesian coordinate system x, y, z whose axes are directed along its length, width, and height (see Fig. 1, respectively. Then

$$E_x = -\frac{\partial \varphi}{\partial x}, \quad E_y = 0, \quad E_z = -\frac{\partial \varphi}{\partial z}, \quad \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial z^2} = 0. \tag{11}$$

[It is implied that the electric field is plane and that $E_x = -V_1(t)/L$ on the surface of the upper electrode.] Assuming that the solution of Eqs. (9) and (6) is a plane flow with rectilinear streamlines, one obtains

$$\vec{v} = v_x(z,t)\vec{i}, \quad p = G(t)x + p_0, \tag{12}$$

$$\rho \frac{\partial v_x}{\partial t} = -G(t) + \eta \frac{\partial^2 v_x}{\partial z^2}, \tag{13}$$

where $\vec{i}$ is the unit vector directed along the x axis; $p_0 = p_{01}$, $G(t) = G_1(t)$ for $-h_1 < z < 0$; and $p_0 = p_{02}$, $G(t) = G_2(t)$ for $0 < z < h_2$. Note that the solution of the linear equation (13) forms an exact solution of the Navier-Stokes equation (9) that is nonlinear in the general case. This is a property of flows of incompressible fluids with rectilinear streamlines, for which the nonlinear terms vanish.

B. Boundary conditions

The boundary conditions include [4] the continuity conditions for the tangential component of the electric intensity,

$$\lim_{z \to +0} \frac{\partial \varphi}{\partial x} = \lim_{z \to -0} \frac{\partial \varphi}{\partial x}, \quad \lim_{z \to h_2 - 0} \frac{\partial \varphi}{\partial x} = \frac{V_1(t)}{L}, \tag{14}$$

the condition for the jump of the normal component of the electric induction,

$$-\varepsilon_2 \lim_{z \to +0} \frac{\partial \varphi}{\partial z} + \varepsilon_1 \lim_{z \to -0} \frac{\partial \varphi}{\partial z} = 4\pi q_{\mathrm{s}}(t), \tag{15}$$

where $q_{\mathrm{s}}(t)$ is the charge surface density, the condition for the jump of the normal component of the electric current density,

$$-\lambda_2 \lim_{z \to +0} \frac{\partial \varphi}{\partial z} + \lambda_1 \lim_{z \to -0} \frac{\partial \varphi}{\partial z} = -\frac{\partial q_{\mathrm{s}}}{\partial t}, \tag{16}$$

no-slip conditions on the walls of the channel,

$$v_x(-h_1, t) = 0, \quad v_x(h_2, t) = 0, \tag{17}$$

no-slip conditions on the interface between the liquids,

$$\lim_{z \to -0} v_x(z,t) = \lim_{z \to +0} v_x(z,t), \tag{18}$$

and the conditions for the jumps of the normal and tangential components of the stress vector,

$$-\lim_{z \to +0} p + \frac{1}{8\pi} \varepsilon_2 \lim_{z \to +0} (E_z^2 - E_x^2) = -\lim_{z \to -0} p + \frac{1}{8\pi} \varepsilon_1 \lim_{z \to -0} (E_z^2 - E_x^2), \tag{19}$$

$$\eta_2 \lim_{z \to +0} \frac{\partial v_x}{\partial z} + \frac{1}{4\pi} \varepsilon_2 \lim_{z \to +0} (E_x E_z) = \eta_1 \lim_{z \to -0} \frac{\partial v_x}{\partial z} + \frac{1}{4\pi} \varepsilon_1 \lim_{z \to -0} (E_x E_z). \tag{20}$$

The boundary condition (20) can be satisfied for a plane flow with rectilinear streamlines only if

$$G_1(t) = G_2(t). \tag{21}$$

The pressure gradient $G(t)=G_1(t)=G_2(t)$ is determined by the hydrodynamic resistance of the external channels through which the liquids are conveyed to and from the considered channel and is regarded for this problem as given in the form

$$G(t) = G_0 + \Re(\tilde{G}_0 e^{2i\omega t}), \tag{22}$$

where G_0 and $\tilde{G}_0$ are some given real and complex numbers, respectively. Here and in what follows, i is the imaginary unit, $\Re$ denotes the real part of a complex expression, and complex quantities are marked with tilde.

With the use of Eqs. (14), (15), and (21), the boundary conditions (19) and (20) can be written in the form

$$p_{02} - p_{01} = \frac{1}{8\pi}\varepsilon_2 \lim_{z\to+0}(E_z^2 - E_x^2) - \frac{1}{8\pi}\varepsilon_1 \lim_{z\to-0}(E_z^2 - E_x^2) = f_{sz}(t), \tag{23}$$

$$\eta_1 \lim_{z\to-0}\frac{\partial v_x}{\partial z} - \eta_2 \lim_{z\to+0}\frac{\partial v_x}{\partial z} = q_s(t)\lim_{z\to0} E_x = f_{sx}(t), \tag{24}$$

where $f_{sx}(t)$ and $f_{sz}(t)$ may be regarded as the surface densities of the tangential and normal components of the Coulomb force. Since the solutions for steady-state oscillatory flows are to be sought for, there is no necessity to use initial conditions.

III. SOLUTION

A. Electric field and charge surface densities

Equation (11) with boundary conditions (14)–(16) for the electric field is independent and can be solved separately. The solution is as follows:

$$\varphi = \varphi_0 - E_x x - E_z z, \tag{25}$$

$$E_x = -\frac{V_l(t)}{L}, \tag{26}$$

$$E_z = \begin{cases} \dfrac{-\varepsilon_1 V_t(t) + 4\pi q_s(t) h_1}{\varepsilon_2 h_1 + \varepsilon_1 h_2} & \text{if } -h_1 \le z < 0 \\ \dfrac{-\varepsilon_2 V_t(t) - 4\pi q_s(t) h_2}{\varepsilon_2 h_1 + \varepsilon_1 h_2} & \text{if } 0 < z \le h_2, \end{cases} \tag{27}$$

$$q_s(t) = \Re(\tilde{q}_s e^{i\omega t}), \tag{28}$$

$$\tilde{q}_s = \frac{V_{la}}{4\pi}\frac{\lambda_2\varepsilon_1 - \lambda_1\varepsilon_2}{\lambda_2 h_1 + \lambda_1 h_2}\frac{1 - i\dfrac{\omega}{\omega_e}}{1 + \dfrac{\omega^2}{\omega_e^2}}, \tag{29}$$

$$\omega_e = 4\pi\frac{\lambda_2 h_1 + \lambda_1 h_2}{\varepsilon_2 h_1 + \varepsilon_1 h_2}, \tag{30}$$

and, in fact, presents the steady-state response of Maxwell's capacitor to an applied harmonically oscillating voltage (see [5], Sec. 7.9). Using Eqs. (26)–(29), one obtains

$$f_{sz}(t) = \frac{\varepsilon_2 - \varepsilon_1}{8\pi}\frac{V_l^2(t)}{L^2} + \frac{\varepsilon_2^3 - \varepsilon_1^3}{8\pi}\frac{V_t^2(t)}{(\varepsilon_2 h_1 + \varepsilon_1 h_2)^2} + \frac{\varepsilon_2^2 h_2 + \varepsilon_1^2 h_1}{(\varepsilon_2 h_1 + \varepsilon_1 h_2)^2} q_s(t) V_t(t) + 2\pi\frac{\varepsilon_2 h_2^2 + \varepsilon_1 h_1^2}{(\varepsilon_2 h_1 + \varepsilon_1 h_2)^2} q_s^2(t), \tag{31}$$

$$f_{sx}(t) = f_{s0x} + \Re(\tilde{f}_{sx} e^{2i\omega t}), \tag{32}$$

$$f_{s0x} = \frac{V_{la}V_{la}}{8\pi L}\frac{\lambda_2\varepsilon_1 - \lambda_1\varepsilon_2}{\lambda_2 h_1 + \lambda_1 h_2}\left(1 + \frac{\omega^2}{\omega_e^2}\right)^{-1}, \tag{33}$$

$$\tilde{f}_{sx} = \frac{V_{la}V_{la}}{8\pi L}\frac{\lambda_2\varepsilon_1 - \lambda_1\varepsilon_2}{\lambda_2 h_1 + \lambda_1 h_2}\frac{1 - i\dfrac{\omega}{\omega_e}}{1 + \dfrac{\omega^2}{\omega_e^2}}. \tag{34}$$

The surface densities of the charge and of the tangential component of the Coulomb force can be written in the form

$$q_s(t) = \frac{V_{la}}{4\pi}\frac{\lambda_2\varepsilon_1 - \lambda_1\varepsilon_2}{\lambda_2 h_1 + \lambda_1 h_2}\frac{\cos\left(\omega t - \arctan\dfrac{\omega}{\omega_e}\right)}{\sqrt{1 + \dfrac{\omega^2}{\omega_e^2}}}, \tag{35}$$

$$f_{sx}(t) = \frac{V_{la}V_{la}}{4\pi L}\frac{\lambda_2\varepsilon_1 - \lambda_1\varepsilon_2}{\lambda_2 h_1 + \lambda_1 h_2}\left(1 + \frac{\omega^2}{\omega_e^2}\right)^{-1/2} \times \cos\left(\omega t - \arctan\frac{\omega}{\omega_e}\right)\cos\omega t. \tag{36}$$

B. Pressure and velocity

Substituting Eqs. (32) and (33) into Eq. (24) and solving Eq. (13) with the boundary conditions (17), (18), and (24), one obtains the following solution for steady-state oscillatory flows:

$$p(x,z,t) = G_0 x + \Re(\tilde{G}e^{2i\omega t})x + p_0(z,t), \tag{37}$$

$$v_x(z,t) = v_{0x}(z) + \Re[\tilde{v}_x(z)e^{2i\omega t}], \tag{38}$$

where

$$p_0(z,t) = \begin{cases} p_{01} & \text{if } -h_1 \le z < 0 \\ p_{01} + f_{sz}(t) & \text{if } 0 < z \le h_2, \end{cases} \tag{39}$$

$$v_{0x}(z) = \begin{cases} \dfrac{h_1 + z}{h_1}\left(v_{s0x} + \dfrac{G_0 z h_1}{2\eta_1}\right) & \text{if } -h_1 \le z < 0 \\ \dfrac{h_2 - z}{h_2}\left(v_{s0x} - \dfrac{G_0 z h_2}{2\eta_2}\right) & \text{if } 0 \le z \le h_2, \end{cases} \tag{40}$$

$$\tilde{v}_x(z) = \begin{cases} \tilde{v}_{sx} \dfrac{\sinh\dfrac{h_1+z}{\tilde{\delta}_1}}{\sinh\dfrac{h_1}{\tilde{\delta}_1}} + 2\sinh\dfrac{z}{2\tilde{\delta}_1}\dfrac{\sinh\dfrac{h_1+z}{2\tilde{\delta}_1}}{\cosh\dfrac{h_1}{2\tilde{\delta}_1}}\dfrac{\tilde{\delta}_1^2}{\eta_1}\tilde{G} & \text{if} -h_1 \le z < 0 \\ \tilde{v}_{sx} \dfrac{\sinh\dfrac{h_2-z}{\tilde{\delta}_2}}{\sinh\dfrac{h_2}{\tilde{\delta}_2}} - 2\sinh\dfrac{z}{2\tilde{\delta}_2}\dfrac{\sinh\dfrac{h_2-z}{2\tilde{\delta}_2}}{\cosh\dfrac{h_2}{2\tilde{\delta}_2}}\dfrac{\tilde{\delta}_2^2}{\eta_2}\tilde{G} & \text{if } 0 \le z \le h_2, \end{cases} \tag{41}$$

$$v_{s0x} = v_{0x}(0) = \frac{f_{s0x} - \dfrac{h_1 G_0}{2} - \dfrac{h_2 G_0}{2}}{\dfrac{\eta_1}{h_1} + \dfrac{\eta_2}{h_2}}, \tag{42}$$

$$\tilde{v}_{sx} = \tilde{v}_x(0) = \frac{\tilde{f}_{sx} - \tanh\dfrac{h_1}{2\tilde{\delta}_1}\tilde{\delta}_1\tilde{G} - \tanh\dfrac{h_2}{2\tilde{\delta}_2}\tilde{\delta}_2\tilde{G}}{\dfrac{\eta_1}{\dfrac{\tilde{\delta}_1}{h_1}\tanh\dfrac{h_1}{\tilde{\delta}_1}} + \dfrac{\eta_2}{\dfrac{\tilde{\delta}_2}{h_2}\tanh\dfrac{h_2}{\tilde{\delta}_2}}}, \tag{43}$$

$$\tilde{\delta}_1 = \frac{\sqrt{2}\delta_1}{1+i} = \frac{(1-i)\delta_1}{\sqrt{2}} = e^{-(\pi/4)i}\delta_1, \quad \delta_1 = \sqrt{\frac{\eta_1}{2\rho_1\omega}}, \tag{44}$$

$$\tilde{\delta}_2 = \frac{\sqrt{2}\delta_2}{1+i} = \frac{(1-i)\delta_2}{\sqrt{2}} = e^{-(\pi/4)i}\delta_2, \quad \delta_2 = \sqrt{\frac{\eta_2}{2\rho_2\omega}}. \tag{45}$$

Here, p_{01} is the pressure at the point $x=0$, $z=-h_1$ regarded as given for this problem.

C. Flow rates of the liquids

Integrating the velocity over the cross sections of the layers, one obtains the flow rates of the liquids,

$$Q_1(t) = Q_{01} + \Re(\tilde{Q}_1 e^{2i\omega t}), \tag{46}$$

$$Q_2(t) = Q_{02} + \Re(\tilde{Q}_2 e^{2i\omega t}), \tag{47}$$

where

$$Q_{01} = \frac{h_1 D v_{s0x}}{2} - \frac{h_1^3 D G_0}{12\eta_1}, \tag{48}$$

$$Q_{02} = \frac{h_2 D v_{s0x}}{2} - \frac{h_2^3 D G_0}{12\eta_2}, \tag{49}$$

$$\tilde{Q}_1 = \tilde{\delta}_1 \tanh\frac{h_1}{2\tilde{\delta}_1} D\tilde{v}_{sx} + \left(2\tilde{\delta}_1 \tanh\frac{h_1}{2\tilde{\delta}_1} - h_1\right)\frac{\tilde{\delta}_1^2}{\eta_1} D\tilde{G}, \tag{50}$$

$$\tilde{Q}_2 = \tilde{\delta}_2 \tanh\frac{h_2}{2\tilde{\delta}_2} D\tilde{v}_{sx} + \left(2\tilde{\delta}_2 \tanh\frac{h_2}{2\tilde{\delta}_2} - h_2\right)\frac{\tilde{\delta}_2^2}{\eta_2} D\tilde{G}. \tag{51}$$

D. Limiting cases

The case $\omega \to 0$ corresponds to a steady flow in a constant electric field. The electric potential and the intensity vector for this case are determined by Eqs. (25)–(27) in which the surface charge density takes the form

$$q_s = \frac{V_{la}}{4\pi}\frac{\lambda_2\varepsilon_1 - \lambda_1\varepsilon_2}{\lambda_2 h_1 + \lambda_1 h_2}. \tag{52}$$

The surface density of the tangential component of the Coulomb force, velocity, and flow rates for the case $\omega \to 0$ take the form

$$f_{sx} = \frac{V_{la}V_{la}}{4\pi L}\frac{\lambda_2\varepsilon_1 - \lambda_1\varepsilon_2}{\lambda_2 h_1 + \lambda_1 h_2}, \tag{53}$$

$$v_x(z) = \begin{cases} \dfrac{h_1+z}{h_1}\left(v_{sx} + \dfrac{G_0 z h_1}{\eta_1}\right) & \text{if} -h_1 \le z < 0 \\ \dfrac{h_2-z}{h_2}\left(v_{sx} - \dfrac{G_0 z h_2}{\eta_2}\right) & \text{if } 0 \le z \le h_2, \end{cases} \tag{54}$$

$$v_{sx} = \frac{f_{sx} - h_1 G_0 - h_2 G_0}{\dfrac{\eta_1}{h_1} + \dfrac{\eta_2}{h_2}}, \tag{55}$$

$$Q_1 = h_1 D v_{sx} - \frac{h_1^3 D G_0}{6\eta_1}, \tag{56}$$

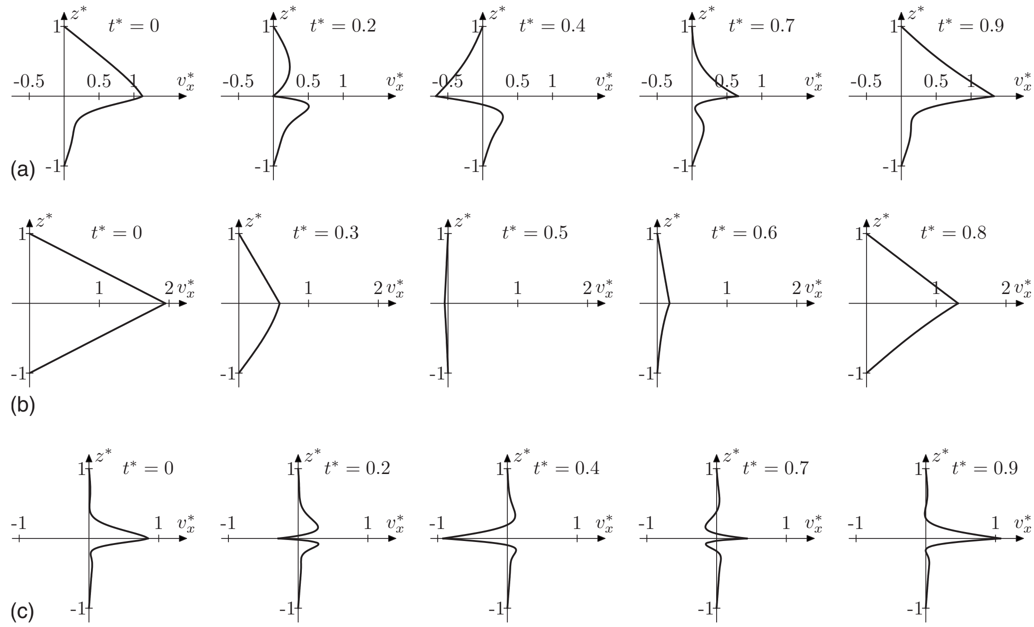

FIG. 2. Evolution of the velocity profile for $\eta_2/\eta_1=1$, $h_2/h_1=1$, $\omega/\omega_e=0.1$, (a) $\delta_1/h_1=0.1$, $\delta_2/h_2=0.5$; (b) $\delta_1/h_1=1$, $\delta_2/h_2=5$, (c) $\delta_1/h_1=0.05$, $\delta_2/h_2=0.1$.

$$Q_2 = h_2 D v_{sx} - \frac{h_2^3 D G_0}{6\eta_2}. \qquad (57)$$

The pressure gradient $G(t)$ is regarded as dependent of the frequency ω and is implied to tend to $2G_0$ as $\omega \to 0$ (i.e., $\widetilde{G} \to G_0$ as $\omega \to 0$). The one-layer EHD flow, which is considered by Melcher and Taylor [4], coincides with the flow in the lower layer with $Q_2=0$ for the limiting case $\lambda_2 \to 0$, $\eta_2 \to 0$.

For $\omega \to \infty$, the following asymptotic relations take place:

$$\widetilde{q}_s \sim e^{-(\pi/2)i} \frac{V_{la}}{\omega_e} \frac{\lambda_2\varepsilon_1 - \lambda_1\varepsilon_2}{\varepsilon_2 h_1 + \varepsilon_1 h_2} \frac{\omega_e}{\omega} \quad \text{as } \omega \to \infty, \qquad (58)$$

$$f_{s0x} \sim \frac{V_{la}V_{la}}{8\pi L} \frac{\lambda_2\varepsilon_1 - \lambda_1\varepsilon_2}{\lambda_2 h_1 + \lambda_1 h_2} \frac{\omega_e^2}{\omega^2} \quad \text{as } \omega \to \infty, \qquad (59)$$

$$\widetilde{f}_{sx} \sim e^{-(\pi/2)i} \frac{V_{la}V_{la}}{8\pi L} \frac{\lambda_2\varepsilon_1 - \lambda_1\varepsilon_2}{\lambda_2 h_1 + \lambda_1 h_2} \frac{\omega_e}{\omega} \quad \text{as } \omega \to \infty, \qquad (60)$$

$$v_{s0x} \sim \frac{V_{la}V_{la}}{8\pi L} \frac{\lambda_2\varepsilon_1 - \lambda_1\varepsilon_2}{\lambda_2 h_1 + \lambda_1 h_2} \left(\frac{\eta_1}{h_1} + \frac{\eta_2}{h_2}\right)^{-1} \frac{\omega_e^2}{\omega^2} \quad \text{as } \omega \to \infty, \qquad (61)$$

$$\widetilde{v}_{sx} \sim e^{-(3\pi/4)i} \frac{V_{la}V_{la}}{8\pi L} \frac{\lambda_2\varepsilon_1 - \lambda_1\varepsilon_2}{\lambda_2 h_1 + \lambda_1 h_2} \times \left(\frac{\eta_1}{h_1}\sqrt{\frac{\omega_e}{\omega_1}} + \frac{\eta_2}{h_2}\sqrt{\frac{\omega_e}{\omega_2}}\right)^{-1} \frac{\omega_e^{3/2}}{\omega^{3/2}} \quad \text{as } \omega \to \infty, \qquad (62)$$

where

$$\omega_1 = \frac{\eta_1}{2\rho_1 h_1^2} = \frac{\omega\delta_1^2}{h_1^2}, \quad \omega_2 = \frac{\eta_2}{2\rho_2 h_2^2} = \frac{\omega\delta_2^2}{h_2^2}. \qquad (63)$$

The asymptotic relations for $\widetilde{v}_{sx}$ and v_{s0x} are written down only for the case $G(t)=0$ since the dependence of the pressure gradient $G(t)$ on the frequency ω is determined by the factors external with respect to this problem.

IV. DISCUSSION

A harmonically oscillating electric field causes oscillatory EHD flow with double frequency in two immiscible liquid layers with a common interface. The normal component of the electric field causes an electric current through the interface. If the ratio of the dielectric permittivities of the liquids does not equal that of electric conductivities, the normal component of the electric current density experiences a jump on the interface, i.e., the ions of conductivity are accumulated on it. The tangential component of the electric field causes the motion of the accumulated ions under the action of the Coulomb force. The motion of the ions actuates the interface and the latter drags the liquids due to viscosity.

The sequences of graphs in Figs. 2(a)–2(c) show the evolution of the velocity profile for various cases during the period of oscillations. [Profiles for the beginning of the period of oscillations, those for the instants when $v_x(0,t)$ reaches maximal and minimal values, and two more intermediate profiles are included into the sequences; more detailed evolution is shown by the movies in the auxiliary material [6].] For all the cases, $G(t)=0$. The dimensionless quantities are determined as follows:

$$z^* = \begin{cases} z/h_1 & \text{if } -h_1 \le z < 0, \\ z/h_2 & \text{if } 0 \le z \le -h_2, \end{cases} \qquad t^* = \frac{\omega t}{\pi}, \quad v_x^* = \frac{v_x}{|\tilde{v}_{sx}|}, \tag{64}$$

providing the dimensionless period of oscillations equal to unity, $-1 \le z^* \le 1$, and $-1 \le v_x^* \le 2$.

The parameters δ_1 and δ_2 have the length dimension and determine the influence of inertia as compared with that of viscosity. If $\delta_1/h_1 \gg 1$ and $\delta_2/h_2 \gg 1$, the quasisteady approximation, within which the influence of inertia is neglected, may be used [see Fig. 2(b)]; otherwise, the influence of inertia should be taken into account. For the case $\delta_1/h_1 \ll 1$, $\delta_2/h_2 \ll 1$, δ_1 and δ_2 are the characteristic thicknesses of the thin layers adjacent to the interface between the liquids within which the velocity has a noticeable oscillating component (the oscillating component of the velocity vanishes outside these thin layers [see Fig. 2(c)]). If $\omega \gg \omega_e$, the maximal value of $|f_{sx}(t)|$ decreases with an increase in the frequency as ω^{-1} [see Eqs. (59) and (60)], i.e., the EHD flow vanishes when the frequency of the applied voltages is sufficiently large.

The investigated phenomenon can be used for pumping in microfluidic devices. A pumped liquid and a liquid that does not mix with it (working liquid) are conveyed to the inlet of a channel with electrodes (similar to that in Fig. 1). After passing the outlet, the working liquid is directed through a special channel to the inlet, and the pumped liquid is conveyed to a necessary destination (hydraulic circuit or network). The obtained solution can be applied to the analysis of processes in such systems including transient processes, in which the voltages are turned on and off. Indeed, since the equations and boundary conditions are linear, the velocity profile for any transient process can be represented in the form of a Fourier integral of the velocity profile for a harmonically oscillating flow.

The electrically charged interface between the liquids is regarded as infinitely thin within the used model. In fact, it is a layer of some finite thickness due to the thermal motion of the ions and molecules. The violation of the bulk electroneutrality within this charged layer does not influence the obtained results if the minimal dimension of the channel is much greater than its thickness, i.e., if the sizes of microchannel cross sections are on the order of micrometer or more. However, the results may change dramatically if this condition is violated. Thus, the miniaturization of microfluidic devices up to nanometer scales requires the further development of the model in order to account for the finite thickness of the nonelectroneutral interface.

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